Meeting 3

Math 22

Section 9.3 The Dot Product

Reading Debrief

- Discuss Preview Activity 9.3.1 w/ group.
- Are there any questions from Sections 9.3.1-9.3.2?

Definition 9.3.4 The dot product of $u = \langle u_1, \ldots, u_n \rangle$ and $v = \langle v_1, \ldots, v_n \rangle$ is the real number

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

∎

Properties of the Dot Product

For all $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$, we have

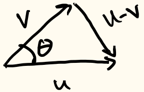
1. $u \cdot v = v \cdot u$
2. $(u+v) \cdot w = u \cdot w + v \cdot w$

3. $(cu)\cdot v = c(u\cdot v)$

4. $u\cdot u = |u|^2$

Section 9.3.2 The Angle Between Vectors

Any two vectors u and v determine a triangle



By Law of Cosines: $|u-v|^2 = |u|^2+|v|^2-2|u||v|\cos\theta$.

OTOH: $|u-v|^2 = (u-v)\cdot(u-v)$

$= |u|^2 - 2u\cdot v + |v|^2$

Equate and cancel: $\boxed{u\cdot v = |u||v|\cos\theta}$

Thus, the angle θ between u and v is

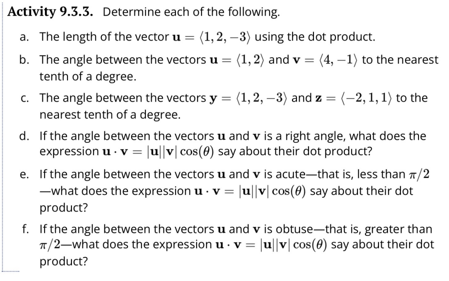
$$\theta = \cos^{-1}\left(\frac{u\cdot v}{|u||v|}\right)$$

Activity 9.3.3

- Complete Activity 9.3.3 and discuss with your group.
- Class discussion.

$u\cdot v = |u||v|\cos(\theta)$

$0 \le \theta \le \pi$



Activity 9.3.3. Determine each of the following.

a. The length of the vector $\mathbf{u} = \langle 1, 2, -3 \rangle$ using the dot product.

b. The angle between the vectors $\mathbf{u} = \langle 1, 2 \rangle$ and $\mathbf{v} = \langle 4, -1 \rangle$ to the nearest tenth of a degree.

c. The angle between the vectors $\mathbf{y} = \langle 1, 2, -3 \rangle$ and $\mathbf{z} = \langle -2, 1, 1 \rangle$ to the nearest tenth of a degree.

d. If the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ is a right angle, what does the expression $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta)$ say about their dot product?

e. If the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ is acute—that is, less than $\pi/2$—what does the expression $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta)$ say about their dot product?

f. If the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ is obtuse—that is, greater than $\pi/2$—what does the expression $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta)$ say about their dot product?

(a) $|u|^2 = u \cdot u$

$|u| = \sqrt{14}$

(b)

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u||v|}\right)$$

$$= \cos^{-1}\left(\frac{2}{\sqrt{5}\sqrt{17}}\right)$$

(c) $$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{14}\sqrt{6}}\right)$$

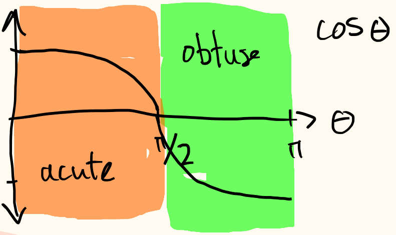
(d) $u \cdot v = 0$ since $\cos(\pi/2) = 0$.

(e) $u \cdot v > 0$ since $\cos(\theta) > 0$ when $0 < \theta < \pi/2$

(f) $u \cdot v < 0$ since $\cos(\theta) < 0$ when $\pi/2 < \theta < \pi$.

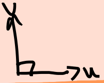
Section 9.3.3 The Dot Product & Orthogonality

The angle θ between u and v satisfies $0 \le \theta \le \pi/2$



$u \cdot v = 0$ iff $\cos(\theta) = 0$

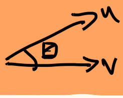
iff $\theta = \pi/2$ iff $u \perp v$



$u \cdot v > 0$ iff $\cos(\theta) > 0$

iff $0 \leq \theta < \pi/2$

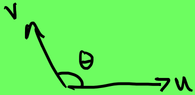
iff θ acute



$u \cdot v < 0$ iff $\cos(\theta) < 0$

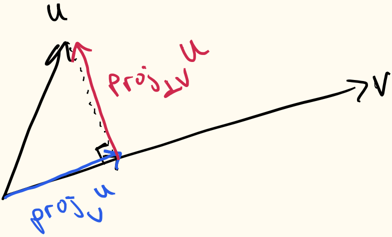
iff $\pi/2 < \theta \leq \pi$

iff θ is obtuse



Section 9.3.5 Projections

Let u and v be vectors in $\mathbb{R}^n$.



We want to write u as a sum of a vector parallel to v and a vector $\perp$ to v:

$$u = \mathrm{proj}_v u + \mathrm{proj}_{\perp v} u$$

We will find a formula for $\mathrm{proj}_v u$.

Since $\mathrm{proj}_v u$ is parallel to v, there exists a scalar c such that

$$\mathrm{proj}_v u = c \cdot v$$

Also know $\mathrm{proj}_{\perp v} u = u - \mathrm{proj}_v u$ is orthogonal to v.

thus,

$$\begin{aligned} 0 &= \mathrm{proj}_{\perp v} u \cdot v \\ &= (u - \mathrm{proj}_v u) \cdot v \\ &= u \cdot v - \mathrm{proj}_v u \cdot v \\ &= u \cdot v - (c \cdot v) \cdot v \\ &= u \cdot v - c(v \cdot v) \end{aligned}$$

If $v \neq 0$, we can solve for c:

$$c = \frac{u \cdot v}{v \cdot v} = \frac{u \cdot v}{|v|^2}$$

Thus,

$$\text{proj}_v u = \left(\frac{u\cdot v}{v\cdot v}\right) v = \frac{u\cdot v}{|v|^2} v$$

$$= \left(\frac{u\cdot v}{|v|}\right)\left(\frac{v}{|v|}\right)$$

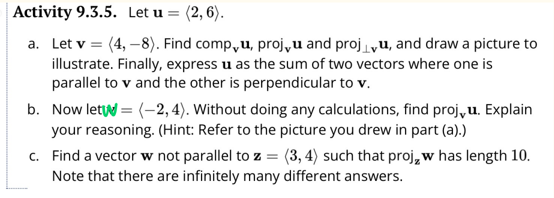
$$\text{proj}_{\perp v} u = u - \text{proj}_v u$$

$$\text{comp}_v u = \frac{u\cdot v}{|v|}$$

The component of u along v ($\text{comp}_v u$) is a scalar which measures the signed magnitude of $\text{proj}_v u$.

Activity 9.3.5

- Complete Activity 9.3.5 and discuss with your group.
- Class discussion.



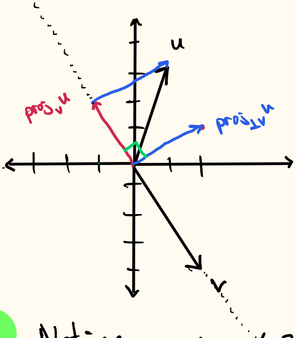
Activity 9.3.5. Let $\mathbf{u} = \langle 2, 6\rangle$.

a. Let $\mathbf{v} = \langle 4, -8\rangle$. Find $\text{comp}_{\mathbf{v}}\mathbf{u}$, $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\text{proj}_{\perp\mathbf{v}}\mathbf{u}$, and draw a picture to illustrate. Finally, express $\mathbf{u}$ as the sum of two vectors where one is parallel to $\mathbf{v}$ and the other is perpendicular to $\mathbf{v}$.

b. Now let $\mathbf{w} = \langle -2, 4\rangle$. Without doing any calculations, find $\text{proj}_{\mathbf{w}}\mathbf{u}$. Explain your reasoning. (Hint: Refer to the picture you drew in part (a).)

c. Find a vector $\mathbf{w}$ not parallel to $\mathbf{z} = \langle 3, 4\rangle$ such that $\text{proj}_{\mathbf{z}}\mathbf{w}$ has length 10. Note that there are infinitely many different answers.

(a)



$\text{comp}_v u = \frac{-40}{\sqrt{80}}$

$\text{proj}_v u = \langle -2, 4 \rangle$

$u - \text{proj}_v u = \langle 2, 6 \rangle - \langle -2, 4 \rangle$
$= \langle 4, 2 \rangle$

(b) Notice $w = \langle -2, 4 \rangle = \text{proj}_v u$.
Since w is parallel to v, $\text{proj}_w u = \text{proj}_v u$.

(c) Skip.

End of Section 9.3

Section 9.4 The Cross Product

Section 9.4.1 Computing the Cross Product

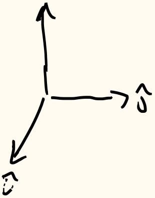
The cross product is defined on the standard unit vectors by the right hand rule

K

$\hat{i}\times\hat{i}=0 \quad \hat{j}\times\hat{j}=0 \quad k\times k=0$

$\boxed{i\times j=k} \quad \boxed{j\times k=\hat{i}} \quad \boxed{k\times i=\hat{j}}$

$\boxed{i\times k=-\hat{j}} \quad \boxed{j\times i=-k} \quad \boxed{k\times\hat{j}=-\hat{i}}$



Its also linear in each argument

$$(au+bv)\times w = a(u\times w)+b(v\times w)$$
$$u\times(av+bw) = a(u\times v)+b(u\times w)$$

We can derive a formula for $u\times v$ using these properties.

Write $u = a\hat{i}+b\hat{j}+c\hat{k}$
$v = d\hat{i}+e\hat{j}+f\hat{k}$

Then $u\times v = (a\hat{i}+b\hat{j}+c\hat{k})\times(d\hat{i}+e\hat{j}+f\hat{k})$

$$= \cancel{(ad)(\hat{i}\times\hat{i})} + (ae)(\hat{i}\times\hat{j}) + (af)(\hat{i}\times\hat{k})$$
$$+ (bd)(\hat{j}\times\hat{i}) + \cancel{(be)(\hat{j}\times\hat{j})} + (bf)(\hat{j}\times\hat{k})$$
$$+ (cd)(\hat{k}\times\hat{i}) + (ce)(\hat{k}\times\hat{j}) + \cancel{(cf)(\hat{k}\times\hat{k})}$$
$$= (ae-bd)\hat{k} + (cd-af)\hat{j} + (bf-ce)\hat{i}$$
$$= \langle bf-ce,\ cd-af,\ ae-bd\rangle$$

Definition 9.4.4 The *cross product* of vectors

$u = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ and $v = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is the vector

$$u \times v = (u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}.$$

∎

Determinants The determinant of a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

the determinant of a 3x3 matrix is defined in terms of 2x2 matrices

$$\begin{array}{ccc} + & - & + \\ \boxed{a} & \boxed{b} & \boxed{c} \\ d & e & f \\ g & h & i \end{array} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

- Cover up row/column containing a to get $\begin{vmatrix} e & f \\ h & i \end{vmatrix}$
- Cover up row/column containing b to get $\begin{vmatrix} d & f \\ g & i \end{vmatrix}$
- cover up row/column containing c to get $\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

In class, I referred to this formula as "expansion along the first row".

You can compute the determinate of a 3x3 in 6 different ways by "expansion along" any row or column as long as you follow the sign convention $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

For example, expansion along the 3rd colum

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = +3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} + 9\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

Now we can express uxv using a determinant. If $u = \langle u_1, u_2, u_3 \rangle$
$v = \langle v_1, v_2, v_3 \rangle$ \

$$u \times v = (u_2v_3 - u_3v_2)\hat{\imath} - (u_1v_3 - u_3v_1)\hat{\jmath} + (u_1v_2 - u_2v_1)\hat{k}.$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\hat{\imath} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\hat{k}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

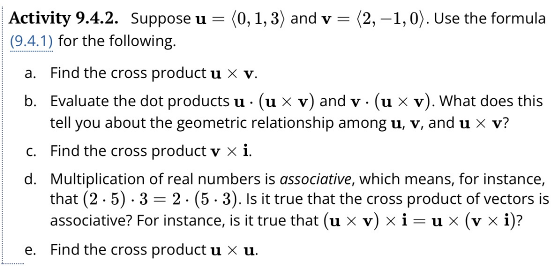
Determinant Formula for uxv

If $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$, then

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Activity 9.4.2

- Complete and discuss w/ your group.
- Class discussion



Activity 9.4.2. Suppose $\mathbf{u} = \langle 0, 1, 3 \rangle$ and $\mathbf{v} = \langle 2, -1, 0 \rangle$. Use the formula (9.4.1) for the following.

a. Find the cross product $\mathbf{u} \times \mathbf{v}$.

b. Evaluate the dot products $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$. What does this tell you about the geometric relationship among $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$?

c. Find the cross product $\mathbf{v} \times \mathbf{i}$.

d. Multiplication of real numbers is *associative*, which means, for instance, that $(2 \cdot 5) \cdot 3 = 2 \cdot (5 \cdot 3)$. Is it true that the cross product of vectors is associative? For instance, is it true that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{i} = \mathbf{u} \times (\mathbf{v} \times \mathbf{i})$?

e. Find the cross product $\mathbf{u} \times \mathbf{u}$.

(a) $$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 0 & 1 & 3 \\ 2 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} \hat{i} - \begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} \hat{j} + \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} \hat{k}$$

$$= \langle 3, 6, -2 \rangle$$

(b) $$u \cdot (u \times v) = \langle 0, 1, 3 \rangle \cdot \langle 3, 6, -2 \rangle = 0$$

$$v\cdot(u\times v) = \langle 2,-1,0\rangle \cdot \langle 3,6,-2\rangle = 0$$

(c) $$v\times i = \begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \overset{+}{\hat{k}} \\ \overset{-}{2} & \overset{+}{-1} & \overset{-}{0} \\ \overset{+}{1} & \overset{-}{0} & \overset{+}{0} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix}\hat{k}$$

$$= \hat{k} = \langle 0,0,1\rangle$$

(d) Not associative!

(e) $$u\times u = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{vmatrix} = \langle 0,0,0\rangle$$

Determinant of any matrix w/ two equal rows (or columns) is 0.

Properties of the Cross Product

Let $u, v, w \in \mathbb{R}^3$. and let $c\in\mathbb{R}$. Then

1. $u\times v = -(v\times u)$
2. $(u+v)\times w = (u\times w) + (v\times w)$
3. $(cu)\times v = c(u\times v) = u\times(cv)$
4. $u\times v = 0$ if and only if u and v are parallel.
5. The cross product is not associative.

∎

Section 9.4.2 The Length of $u \times v$

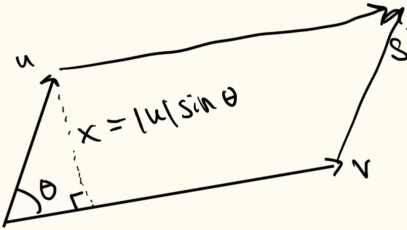
You can compute the length of the cross product directly from the definitions and atleast one trig identity.

The Length of $u \times v$ Let $u, v \in \mathbb{R}^3$. Then

$$|u \times v| = |u||v| \sin\theta$$

where $0 \le \theta \le \pi$ is the angle between u and v.

▨



$$x = |u| \sin\theta$$

$$\sin(\theta) = \frac{x}{|u|}$$

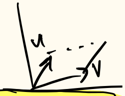
From the picture:

$$\boxed{\begin{aligned}|u \times v| &= |u||v| \sin\theta \\ &= \text{area of parallelogram spanned by } u, v.\end{aligned}}$$

This explains why $|u \times v| = 0 \iff u, v$ are parallel.

Consider $u = \langle u_1, u_2, 0 \rangle$, $v = \langle v_1, v_2, 0 \rangle$

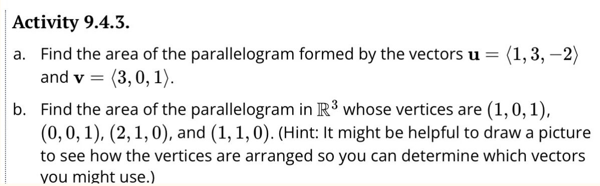
$$|\hat{k}| \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix} = |u \times v| = \text{area of parallegram}$$



Thus, $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$ is the (signed) area of a parallelogram!

Activity 9.4.3

- Complete Activity 9.4.3 and discuss w/ your group.
- Class discussion.



Activity 9.4.3.

a. Find the area of the parallelogram formed by the vectors $\mathbf{u} = \langle 1, 3, -2 \rangle$ and $\mathbf{v} = \langle 3, 0, 1 \rangle$.

b. Find the area of the parallelogram in $\mathbb{R}^3$ whose vertices are $(1,0,1)$, $(0,0,1)$, $(2,1,0)$, and $(1,1,0)$. (Hint: It might be helpful to draw a picture to see how the vertices are arranged so you can determine which vectors you might use.)

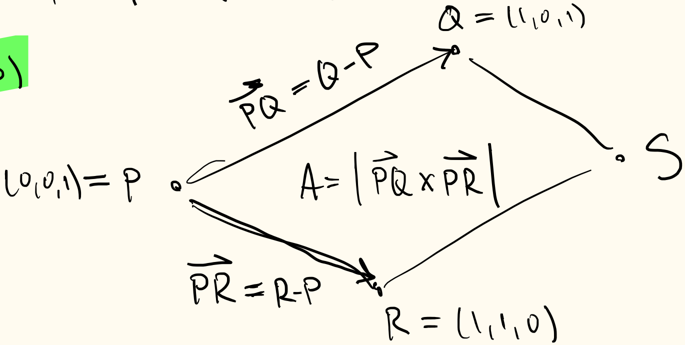
(a) Compute $u \times v$ using determinant formula.

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix} \hat{k}$$

$$= \langle 3, -7, -9 \rangle$$

$$|u \times v| = \sqrt{139}$$

(b)



Q = (1,0,1)
$\overrightarrow{PQ} = Q - P$
(0,0,1) = P
$A = |\overrightarrow{PQ} \times \overrightarrow{PR}|$
S
$\overrightarrow{PR} = R - P$
R = (1,1,0)

$$A = |\overrightarrow{PQ} \times \overrightarrow{PR}| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ {}^{-}1 & {}^{+}0 & {}^{-1}0 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \left| -1 \begin{vmatrix} \hat{j} & \hat{k} \\ 1 & -1 \end{vmatrix} \right|$$

$$= |(-\hat{j} - k)|$$

$$= |\langle 0, -1, -1 \rangle| = \sqrt{2}.$$